

## On intertwining dilations. V

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*Dedicated to the 65<sup>th</sup> anniversary of Professor Béla Sz.-Nagy*

**Introduction.** The interest of a functional labelling of the intertwining dilations<sup>1)</sup> of a given contraction  $A$  intertwining two contractions  $T'$  and  $T$  (i.e.  $T'A=AT$ ) was stressed in [18], where such a labelling, involving analytic and non-analytic operator-valued functions, was used in the study of some pure operator theory questions. More recently, in [11], a functional labelling, by means of contractive analytic operator-valued functions, was shown to play a central role in an electrical engineering problem, in the case when  $T'=T$  are contractions of class  $C_0(N)$  (in the sense of [16], Ch. IX, Sec. 3). However, in the cases  $T'=S^*$ ,  $T=S$ , where  $S$  is a Jordan operator (on a finite dimensional space) or a unilateral shift, this kind of labelling was already obtained by SCHUR (implicitly, for the numerical case, in his classical research on extrapolation [14]) and by ADAMIAN—AROV—KREIN (explicitly, for the operatorial case, in their basic research on Hankel operators [1], [2], [3], [4]).

The general case (considered for instance in [17], [16], [10], [8], [5], etc.), namely for arbitrary contractions  $A$ ,  $T'$ ,  $T$  and arbitrary contractive (but not necessarily strictly contractive) intertwining dilations, seems to have not been considered. The first aim of this paper is to fill this gap by showing that in this most general case there exists also a labelling by contractive analytic operator valued functions. This labelling was suggested, by the previous papers [6], [9], [7].

In establishing this labelling (in Sec. 4 below) we shall establish a new one. Namely we shall show that the contractive intertwining dilations can be labelled by sequences  $\{\Gamma_n\}_1^\infty$  of contractions such that  $\Gamma_1$  acts between two suitable spaces while, for  $n \geq 2$ ,  $\Gamma_n$  acts between the closed ranges of  $I - \Gamma_{n-1}^* \Gamma_{n-1}$  and  $I - \Gamma_{n-1} \Gamma_{n-1}^*$  (see Sec. 3 below). This labelling was imposed to us by a problem in geophysics

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<sup>1)</sup> For the terminology and partly for notations, which are essentially those of [16], [8], [6], see the next Section 1.

(where the  $\Gamma_n$ 's have a concrete physical meaning) and by its numerical treatment. These connections will be discussed elsewhere. However, in Sec. 5 we give an application of our results to the classification of ANDO's isometric dilations of a pair of commuting contractions [5].

Finally, let us remark that at this stage of our research the explicit connection of this paper with [18] is still an open (and seemingly, basic) question.

Also, we take this opportunity to thank our colleague Gr. Arsene for the useful discussions on the subject of this Note.

1. We start by giving the main notations and recalling some basic facts concerning contractive intertwining dilations.

Let  $\mathfrak{H}$  and  $\mathfrak{H}'$  be some Hilbert spaces<sup>2)</sup> and let  $L(\mathfrak{H}, \mathfrak{H}')$  denote the algebra of all operators from  $\mathfrak{H}$  to  $\mathfrak{H}'$ ; in case  $\mathfrak{H}=\mathfrak{H}'$ ,  $L(\mathfrak{H}, \mathfrak{H})$  will be denoted simply by  $L(\mathfrak{H})$ . For two contractions,  $T \in L(\mathfrak{H})$ ,  $T' \in L(\mathfrak{H}')$  we denote by  $\mathcal{J}(T', T)$  the set of the  $A \in L(\mathfrak{H}, \mathfrak{H}')$  intertwining  $T'$  and  $T$ , i.e. such that  $T'A = AT$ . Let  $U \in L(\mathfrak{R})$ ,  $U' \in L(\mathfrak{R}')$  be the minimal isometric dilations of  $T$  and  $T'$ , respectively; for  $n \geq 0$  let  $P_n, P'_n$  denote the orthogonal projections of  $\mathfrak{R}$  and  $\mathfrak{R}'$  onto

$$\mathfrak{H}_n = \begin{cases} \mathfrak{H} \\ \mathfrak{H} + \mathfrak{Q} + U\mathfrak{Q} + \dots + U^{n-1}\mathfrak{Q} \end{cases} \quad \text{and} \quad \mathfrak{H}'_n = \begin{cases} \mathfrak{H}' \\ \mathfrak{H}' + \mathfrak{Q}' + U'\mathfrak{Q}' + \dots + U'^{n-1}\mathfrak{Q}' \end{cases} \quad \begin{matrix} (n=0) \\ (n \geq 1), \end{matrix}$$

respectively, where  $\mathfrak{Q} = ((U-T)\mathfrak{H})^\perp$ ,  $\mathfrak{Q}' = ((U'-T')\mathfrak{H}')^\perp$ . We also set  $P = P_0$ ,  $P' = P'_0$  and

$$T_n = P_n U|_{\mathfrak{H}_n}, \quad T'_n = P'_n U'|_{\mathfrak{H}'_n} \quad (n = 0, 1, 2, \dots);$$

obviously  $T_0 = T$ ,  $T'_0 = T'$  and  $U, U'$  are also minimal isometric dilations of  $T_N, T'_N$ , respectively ( $N=0, 1, 2, \dots$ ). In the sequel  $A$  will be a contraction  $\in \mathcal{J}(T', T)$ .

By a *contractive intertwining dilation* (CID), or an  $N^{\text{th}}$  *partial contractive intertwining dilation* (N-PCID) of  $A$  we respectively mean operators  $A_\infty \in L(\mathfrak{R}, \mathfrak{R}')$  and  $A_N \in L(\mathfrak{H}_N, \mathfrak{H}'_N)$  such that

$$(1.1) \quad \|A_\infty\| \leq 1, \quad A_\infty \in \mathcal{J}(U', U), \quad P' A_\infty = A P,$$

$$(1.1)_N \quad \|A_N\| \leq 1, \quad A_N \in \mathcal{J}(T'_N, T_N), \quad P' A_N = A(P|_{\mathfrak{H}_N}).$$

Thus, the operator

$$(1.2)_n \quad A_n = P'_n A_v|_{\mathfrak{H}_n}, \quad \text{where} \quad \begin{matrix} n = 0, 1, 2, \dots & \text{if } v = \infty, \text{ and} \\ n = 0, 1, 2, \dots, N & \text{if } v = N, \end{matrix}$$

<sup>2)</sup> Hilbert spaces will be considered complex and their subspaces, if not specified, will be assumed to be linear and closed. Operators will always be assumed to be linear and bounded; also when confusion might occur the identity operator  $I$  and the null operator  $O$  on a Hilbert space  $\mathfrak{G}$  will be denoted by  $I_{\mathfrak{G}}$  and  $O_{\mathfrak{G}}$ , respectively.

is an  $n$ -PCID of  $A$ , and

$$(1.3)_n \quad P'_n A_{n+1} = A_n(P_n|_{\mathfrak{H}_{n+1}}) \quad \text{for } 0 \leq n < v;$$

moreover, in the first case we have

$$(1.4) \quad A_\infty = \text{strong lim } A_n P_n \quad (n \rightarrow \infty).$$

It is also easy to verify that, conversely, if a sequence of  $n$ -PCID's  $A_n$  satisfies conditions (1.3)<sub>n</sub> ( $n=0, 1, 2, \dots$ ), then the strong limit in (1.4) exists and defines a CID of  $A$ . Therefore we can make the following

**Remark 1.1.** There exists a one-to-one correspondence (given by (1.2)<sub>n</sub> and (1.4)) between the CID's of  $A$  and the sequences  $\{A_n\}_0^\infty$  of  $n$ -PCID's of  $A$ ,  $A_n$  satisfying (1.3)<sub>n</sub> ( $n=0, 1, 2, \dots$ ).

In order to facilitate the exposition, we shall give several useful facts, which actually resume the original construction of a CID (see [17], [10], [16], [7]). To this aim, let  $T$ ,  $T'$  and  $A$  be as above. We set<sup>3)</sup>

$$(1.5) \quad \mathfrak{F}_A = \{D_A T h + (U - T)h : h \in \mathfrak{H}\}^-, \quad \mathfrak{R}_A = (\mathfrak{D}_A \oplus \mathfrak{L}) \ominus \mathfrak{F}_A$$

and

$$(1.5)' \quad \mathfrak{F}'_A = \{D_A h \oplus (U' - T')Ah : h \in \mathfrak{H}\}^-, \quad \mathfrak{R}'_A = (\mathfrak{D}_A \oplus \mathfrak{L}') \ominus \mathfrak{F}'_A.$$

**Lemma 1.1.** Let  $T$ ,  $T'$  and  $A$  be as above. Then

$$(1.6) \quad C(D_A T h + (U - T)h) = (U' - T')Ah \quad (h \in \mathfrak{H})$$

defines a contraction  $C = C_A \in L(\mathfrak{F}_A, \mathfrak{L}')$ . Moreover, the formula

$$(1.7) \quad W_A D_{C_A}^* l' = R'_A (0_{\mathfrak{D}_A} \oplus l') \quad (l' \in \mathfrak{L}'),$$

where  $R'_A$  denotes the orthogonal projection of  $\mathfrak{D}_A \oplus \mathfrak{L}'$  onto  $\mathfrak{R}'_A$ , defines a unitary operator from  $\mathfrak{D}_{C_A}^*$  onto  $\mathfrak{R}'_A$ .

**Proof.** Let  $i_{\mathfrak{L}'}$  and  $\omega$  be the operators defined by

$$i_{\mathfrak{L}'}(l') = 0_{\mathfrak{D}_A} \oplus l' \in \mathfrak{D}_A \oplus \mathfrak{L}' \quad (l' \in \mathfrak{L}'),$$

$$\omega(D_A T h + (U - T)h) = D_A h \oplus (U' - T')Ah \quad (h \in \mathfrak{H}).$$

Obviously  $i_{\mathfrak{L}'}$  is unitary from  $\mathfrak{L}'$  to  $\{0\} \oplus \mathfrak{L}' \subset \mathfrak{D}_A \oplus \mathfrak{L}'$ ; also,  $\omega$  is unitary from  $\mathfrak{F}_A$  to  $\mathfrak{F}'_A$  since, by virtue of [16], Sec. II. 1, we have

$$\begin{aligned} \|D_A T h + (U - T)h\|^2 &= \|D_A T h\|^2 + \|(U - T)h\|^2 = \|T h\|^2 - \|A T h\|^2 + \|D_T h\|^2 = \\ &= \|h\|^2 - \|A T h\|^2 = \|D_A h\|^2 + \|A h\|^2 - \|T' A h\|^2 = \|D_A h\|^2 + \|D_{T'} A h\|^2 = \\ &= \|D_A h\|^2 + \|(U' - T')A h\|^2 = \|D_A h \oplus (U' - T')A h\|^2 \end{aligned}$$

<sup>3)</sup> Recall that for any operator  $C$  from a Hilbert space  $\mathfrak{G}$  to another one  $\mathfrak{G}'$ ,  $D_C$  denotes the defect operator  $((I - C^* C)^2)^{1/4}$  and  $\mathfrak{D}_C = (D_C \mathfrak{G})^-$ ; if  $\|C\| \leq 1$ , then obviously  $D_C = (I - C^* C)^{1/2}$ .

for all  $h \in \mathfrak{H}$ . We shall consider  $i_{\mathfrak{L}'}$  as operator from  $\mathfrak{L}'$  to  $\mathfrak{D}_A \oplus \mathfrak{L}'$  and extend  $\omega$  on the whole of  $\mathfrak{D}_A + \mathfrak{L}$  by setting  $\omega r = 0_{\mathfrak{D}_A \oplus \mathfrak{L}'}$  for  $r \in \mathfrak{R}_A$ . Then  $C_A = i_{\mathfrak{L}'}^* \omega|_{\mathfrak{F}_A}$ ; hence  $C_A$  is a contraction and

$$C_A^* = \omega^* i_{\mathfrak{L}'}, \quad C_A C_A^* = i_{\mathfrak{L}'}^* \omega \omega^* i_{\mathfrak{L}'}, \quad D_{C_A}^2 = i_{\mathfrak{L}'}^* R'_A i_{\mathfrak{L}'}.$$

It follows that

$$\|D_{C_A^*} l'\|^2 = (D_{C_A^*}^2 l', l') = (i_{\mathfrak{L}'}^* R'_A i_{\mathfrak{L}'} l', l') = \|R'_A i_{\mathfrak{L}'} l'\|^2 = \|R'_A (0_{\mathfrak{D}_A} \oplus l')\|^2 \quad (l' \in \mathfrak{L}')$$

and consequently that  $W_A$  is an isometric operator from  $\mathfrak{D}_{C_A^*}$  to  $\mathfrak{R}'_A$ . If  $d_0 \oplus l'_0 \in \mathfrak{R}'_A$  is orthogonal to the range of  $W_A$  then

$$(l'_0, l') = (d_0 \oplus l'_0, 0 \oplus l') = (d_0 \oplus l'_0, W_A D_{C_A^*} l') = 0 \quad (l' \in \mathfrak{L}')$$

whence  $l'_0 = 0$ . But, by (1.5)',  $(d_0, D_A h) = (d_0 \oplus l'_0, D_A h \oplus (U' - T') Ah) = 0$  ( $h \in \mathfrak{H}$ ), whence  $d_0 = 0$ , since  $d_0 \in \mathfrak{D}_A$ . Thus  $d_0 \oplus l'_0 = 0$  and consequently we conclude that  $W_A$  is unitary.

**Lemma 1.2.** *Let  $T, T'$  and  $A$  be as in Lemma 1.1. Then the formula*

$$(1.8) \quad C(A_1)(D_A P + I - P)|_{\mathfrak{H}_1} = (I - P')A_1$$

*establishes a one-to-one correspondence between the 1-PCID  $A_1$  of  $A$  and all contractions*

$$(1.9) \quad C: \mathfrak{D}_A + \mathfrak{L} \rightarrow \mathfrak{L}', \quad C|_{\mathfrak{F}_A} = C_A.$$

*Moreover, the formula*

$$(1.10) \quad X(A_1)D_{C(A_1)}(D_A P + I - P)|_{\mathfrak{H}_1} = D_{A_1}$$

*defines a unitary operator from  $\mathfrak{D}_{C(A_1)}$  to  $\mathfrak{D}_{A_1}$ .*

**Proof.** Let  $A_1$  be a 1-PCID of  $A$ .

Then, since by (1.1),

$$\begin{aligned} (1.11) \quad & \| (I - P')A_1 h_1 \|^2 = \| A_1 h_1 \|^2 - \| P'A_1 h_1 \|^2 = \\ & = \| A_1 h_1 \|^2 - \| AP h_1 \|^2 \leq \| h_1 \|^2 - \| AP h_1 \|^2 = \| h_1 \|^2 - \| P h_1 \|^2 + \| D_A P h_1 \|^2 = \\ & = \| (I - P) h_1 \|^2 + \| D_A P h_1 \|^2 = \| (D_A P + I - P) h_1 \|^2 \quad (h_1 \in \mathfrak{H}_1), \end{aligned}$$

we infer that formula (1.8) defines a contraction  $C = C(A_1)$  from  $\mathfrak{D}_A + \mathfrak{L}$  to  $\mathfrak{L}' = (I - P)\mathfrak{H}_1$ . Moreover, since  $(D_A P + I - P)T_1 h = D_A T h + (U - T)h$  ( $h \in \mathfrak{H}$ ), we also have

$$\begin{aligned} C(D_A T h + (U - T)h) &= (I - P')A_1 T_1 h = (I - P')T'_1 A_1 h = \\ &= (I - P')T'_1 P' A_1 h = (I - P')T'_1 A h = (U' - T')Ah = \\ &= C_A(D_A T h + (U - T)h) \quad (h \in \mathfrak{H}), \end{aligned}$$

i.e.  $C|\mathfrak{F}_A = C_A$ . Also, from  $A_1 = P'A_1 + (I - P')A_1 = AP|\mathfrak{F}_1 + (I - P')A_1$  we obtain

$$(1.12) \quad A_1 = (AP + C(D_A P + I - P))|\mathfrak{F}_1.$$

This formula shows that  $A_1$  is uniquely determined by  $C = C(A_1)$ . Let now  $C$  be any contraction with the properties (1.9) and let  $A_1 \in L(\mathfrak{F}_1)$  be defined by (1.12). Then, the relation  $P'A_1 = AP|\mathfrak{F}_1$  is obvious and therefore

$$\begin{aligned} T'_1 A_1 h_1 &= T'_1 P'A_1 h_1 = T'_1 A P h_1 = T' A P h_1 + (U' - T') A P h_1 = \\ &= T' A P h_1 + C_A (D_A T + U - T) P h_1 = A T P h_1 + C_A (D_A T + U - T) P h_1 = \\ &= A T P h_1 + C (D_A T + U - T) P h_1 = A P T_1 h_1 + C (D_A P T_1 + (I - P) T_1) h_1 = \\ &= (AP + C(D_A P + I - P)) T_1 h_1 = A_1 T_1 h_1 \quad (h_1 \in \mathfrak{F}_1), \end{aligned}$$

i.e.  $A_1 \in \mathcal{J}(T'_1, T_1)$ . Moreover, from (1.12) we infer

$$(1.13) \quad \|h_1\|^2 - \|A_1 h_1\|^2 = \|h_1\|^2 - \|A P h_1\|^2 - \|C(D_A P + I - P)h_1\|^2 = \\ = \|D_C(D_A P + I - P)h_1\|^2 \quad (h_1 \in \mathfrak{F}_1)$$

This shows that  $A_1$  is a contraction. We have thus verified that  $A_1$  has the properties (1.1)<sub>1</sub>. The last statement of the lemma follows now readily from (1.13).

**Remark 1.2.** The basic existence theorem [17], [16] for a CID  $A_\infty$  of a contraction  $A \in \mathcal{J}(T', T)$ , where  $T', T$  are as above, follows from the preceding lemmas by the following simple recurrent construction.

Set  $A_0 = A$  and set  $C_1 = C_A Q_1$  where  $Q_1$  denotes the orthogonal projection of  $\mathfrak{D}_{A_0} + \mathfrak{L}$  onto  $\mathfrak{F}_{A_0}$ . Define  $A_1$  as the 1-PCID such that  $C(A_1) = C_1$ . Repeat the same procedure with  $A_1$ ,  $U\mathfrak{L}$  and  $U'\mathfrak{L}'$  in the roles of  $A_0$ ,  $\mathfrak{L}$  and  $\mathfrak{L}'$  and obtain  $A_2$ , and so on. Finally one obtains a sequence  $\{A_n\}_{n=0}^\infty$  of  $n$ -PCID's  $A_n$  of  $A$  satisfying the conditions (1.3)<sub>n</sub> ( $n=0, 1, 2, \dots$ ) and consequently a CID  $A_\infty$  of  $A$ , by virtue of Remark 1.1.

**2.** Let  $T, T'$  and  $A, \|A\| \leq 1$ , be as in Sec. 1. Let, moreover,  $A_N$  be an  $N$ -PCID of  $A$  and  $A_n$  be the operator defined by (1.2)<sub>n</sub> ( $n=1, 2, \dots, N-1$ ). From Lemma 1.2 it follows readily that  $A_N$  is uniquely determined by, and also uniquely determines, a string  $\{C_n\}_{n=1}^N$  of  $U'^{n-1}\mathfrak{L}'$ -valued contractions  $C_n$  ( $n=1, 2, \dots, N-1$ ), namely the string  $\{C(A_n)\}_1^N$ . However, the definition of the string  $\{C(A_n)\}_1^N$  explicitly involves, besides the operators  $U, U'$  (i.e. the minimal isometric dilations of  $T, T'$  respectively) and  $A$ , also the operators  $A_1, \dots, A_{N-1}$ . In order to get rid of the explicit reference to  $A_1, \dots, A_{N-1}$  in the characterization of  $A_N$  by a string of  $U'^{n-1}\mathfrak{L}'$ -valued contractions  $C_n$  ( $n=1, 2, \dots, N$ ), we firstly introduce the following:

**Definition 2.1.** A string or a sequence  $\{C_n\}_{1 \leq n < v}$  (where  $v=1, 2, \dots, \infty$ ; in the case  $v=\infty$  we set  $v-1=\infty$ ) of operators

$$(2.1) \quad C_n: \mathfrak{D}_{n-1} + U^{n-1}\mathfrak{Q} \rightarrow U'^{n-1}\mathfrak{Q}' \quad (1 \leq n < v)$$

is called an *A-cascade* if each  $C_n$  ( $1 \leq n \leq v$ ) is a contraction,

$$(2.2) \quad \mathfrak{D}_0 = \mathfrak{D}_A, \quad \mathfrak{D}_n = \mathfrak{D}_{C_n} \quad (1 \leq n < v-1),$$

$$(2.3)_1 \quad C_1|_{\mathfrak{F}_A} = C_A,$$

$$(2.3)_2 \quad C_2(D_{C_1}(D_A Th + (U-T)h) + Ul_1) = U' C_1(D_A h + l_1) \quad (h \in \mathfrak{H}, l_1 \in \mathfrak{Q})$$

(in case  $v > 2$ ), and

$$(2.3)_n \quad C_n(D_{C_{n-1}}(D_{C_{n-2}}(\dots(D_{C_1}(D_A Th + (U-T)h) + Ul_1)\dots) + U^{n-2}l_{n-2}) + U^{n-1}l_{n-1}) = \\ = U' C_{n-1}(D_{C_{n-2}}(\dots(D_{C_1}(D_A h + l_1) + Ul_2)\dots) + U^{n-2}l_{n-1}) \quad (h \in \mathfrak{H}, l_1, \dots, l_{n-1} \in \mathfrak{Q})$$

for all  $3 \leq n < v$  (in case  $v > 3$ ).

In the next two lemmas,  $\{C_n\}_{1 \leq n < v}$  will be any fixed *A-cascade* string or sequence; also the spaces  $\mathfrak{D}_n$  ( $0 \leq n < v$ ) will have the same meaning as in the preceding definition.

**Lemma 2.1.** *There exists a unique string (or sequence, respectively)  $\{Y_n\}_{1 \leq n < v}$  of isometric operators*

$$(2.4) \quad Y_n: \mathfrak{D}_{n-1} \rightarrow \mathfrak{D}_n \quad (1 \leq n < v)$$

such that

$$(2.5)_1 \quad Y_1 D_A h = D_{C_1}(D_A Th + (U-T)h) \quad (h \in \mathfrak{H}),$$

and (if  $v > 2$  then) for  $2 \leq n < v$ :

$$(2.5)_n \quad Y_n D_{C_{n-1}}(d + U^{n-2}l) = D_{C_n}(Y_{n-1}d + U^{n-1}l) \quad (d \in \mathfrak{D}_{n-2}, l \in \mathfrak{Q}).$$

**Proof.** We have, by (2.3)<sub>1</sub>,

$$\begin{aligned} \|D_A h\|^2 &= \|Th\|^2 - \|ATh\|^2 + \|D_T h\|^2 - \|D_{T'} Ah\|^2 = \\ &= \|D_A Th + (U-T)h\|^2 - \|(U'-T')Ah\|^2 = \\ &= \|D_A Th + (U-T)h\|^2 - \|C_A(D_A Th + (U-T)h)\|^2 = \|D_{C_1}(D_A Th + (U-T)h)\|^2 \\ &\quad (h \in \mathfrak{H}), \end{aligned}$$

thus, indeed, (2.5)<sub>1</sub> defines an isometric operator from  $\mathfrak{D}_0 = \mathfrak{D}_A$  to  $\mathfrak{D}_1 = \mathfrak{D}_{C_1}$ . We assume now that (2.5)<sub>n</sub> (for  $n=m-1, 2 \leq m < v$ ) defines an isometric operator  $Y_{m-1}$ , obviously in a unique manner. Then definition (2.3)<sub>m</sub> can be written under the form

$$C_m(Y_{m-1}d + U^{m-1}l) = U' C_{m-1}(d + U^{m-2}l) \quad (d \in \mathfrak{D}_{m-2}, l \in \mathfrak{Q});$$

consequently we have

$$\begin{aligned}
 \|D_{C_{m-1}}(d+U^{m-2}l)\|^2 &= \|d+U^{m-2}l\|^2 - \|C_{m-1}(d+U^{m-2}l)\|^2 = \\
 &= \|d\|^2 + \|U^{m-2}l\|^2 - \|U'C_{m-1}(d+U^{m-2}l)\|^2 = \\
 &= \|Y_{m-1}d\|^2 + \|U^{m-1}l\|^2 - \|C_m(Y_{m-1}d+U^{m-1}l)\|^2 = \\
 &= \|Y_{m-1}d+U^{m-1}l\|^2 - \|C_m(Y_{m-1}d+U^{m-1}l)\|^2 = \|D_{C_m}(Y_{m-1}d+U^{m-1}l)\|^2 \\
 &\quad (d \in \mathfrak{D}_{m-1}, l \in \mathfrak{L}).
 \end{aligned}$$

These relations show that, indeed,  $(2.5)_m$  defines the searched isometric operator  $Y_m$ . Thus the lemma is proved by recurrence.

**Remark 2.1.** By virtue of Lemma 2.1, it is easy to infer that the definitions  $(2.3)_n$  (for  $2 \leq n < v$ ) can be written under the compact form

$$(2.6)_n \quad C_n(Y_{n-1}d+U^{n-1}l) = U'C_{n-1}(d+U^{n-2}l) \quad (d \in \mathfrak{D}_{n-2}, l \in \mathfrak{L}).$$

Also let us notice that

$$(2.7)_n \quad \mathfrak{D}_n \subset \mathfrak{D}_{n-1} + U^{n-1}\mathfrak{L}, \quad \mathfrak{D}_n \subset \mathfrak{H}_n$$

$(1 \leq n < v-1)$  and  $\mathfrak{D}_0 \subset \mathfrak{H}$ .

Now let us consider any isometric operator  $X: \mathfrak{D}_1 \rightarrow \mathfrak{H}_1$ . By  $(2.7)_n$  ( $2 \leq n < v$ ), we can attach to  $X$  the string (or sequence)  $\{X_n\}_{1 \leq n < v}$  of unitary operators

$$(2.8) \quad X_n: \mathfrak{D}_n + U^n\mathfrak{L} \rightarrow \mathfrak{H}_n = \text{Range}(X_n) \quad (1 \leq n < v)$$

by the following recurrent manner:

$$(2.9) \quad \mathfrak{H}_1 = \text{Range}(X) + U\mathfrak{L},$$

$$(2.10)_n \quad X_n|U^n\mathfrak{L} = I_{U^n\mathfrak{L}} \quad (1 \leq n < v),$$

$$(2.11)_1 \quad X_1|\mathfrak{D}_1 = X \quad (2 \leq n < v)$$

and, in case  $v > 2$ ,

$$(2.11)_n \quad X_n|\mathfrak{D}_n = X_{n-1}|\mathfrak{D}_n$$

This sequence will be improperly called the  $\{C_n\}_{1 \leq n < v}$ -extension of  $X$ .

Let, moreover,  $A_1$  be the 1-PCID of  $A$  such that  $C(A_1) = C_1$  (see Lemma 1.2) and let  $\{X_n\}_{1 \leq n < v}$  be the  $\{C_n\}_{1 \leq n < v}$ -extension of  $X(A_1)$ ; then by virtue of (2.1) and (2.8)  $C'_n = C_{n+1}X_n^* \in L(\mathfrak{H}_n, U'^n\mathfrak{L})$  ( $1 \leq n < v-1$ ). In case  $v > 2$ , the string (or sequence)  $\{C'_n\}_{1 \leq n < v-1}$  will be called, for convenience, the *reduced string* (or sequence) of  $\{C_n\}_{1 \leq n < v}$ .

**Lemma 2.2.** *The reduced string (or sequence) is an  $A_1$ -cascade.*

**Proof.** It is plain that  $C'_n$  ( $1 \leq n < v$ ) are contractions from  $\mathfrak{D}'_{n-1} + U^{n-1}(U\mathfrak{L})$  to  $U'^{n-1}(U'\mathfrak{L})$ , where  $\mathfrak{D}'_0 = \mathfrak{D}_{A_1}$  and  $\mathfrak{D}'_{n-1} = X_n\mathfrak{D}_n$  ( $1 \leq n < v$ ). Moreover, since

$C'_n X_n = C_{n+1}$  ( $1 \leq n < v-1$ ), we have  $X_n^* C_n'^* C'_n X_n = C_{n+1}^* C_{n+1}$ ,  $C_n'^* C'_n X_n = X_n C_{n+1}^* C_{n+1}$  whence

$$(2.12)_n \quad D_{C'_n} X_n = X_n D_{C_{n+1}} \quad (1 \leq n < v-1).$$

Since, by (2.11)<sub>2</sub>, (2.1), (2.2), (2.10)<sub>1</sub>, (2.11)<sub>1</sub> and (1.10), we have

$$\mathfrak{D}'_1 = X_2 \mathfrak{D}_2 = X_1 \mathfrak{D}_2 \subset X_1 (\mathfrak{D}_1 + U\mathfrak{L}) = X(A_1) \mathfrak{D}_{C_1} + U\mathfrak{L} = \mathfrak{D}_{A_1} + U\mathfrak{L} = \mathfrak{D}' + U\mathfrak{L}$$

and, since (if  $v > 3$  and  $2 \leq n < v-1$ ) we have, by (2.11)<sub>n+1</sub>, (2.2), (2.12)<sub>n</sub> and (2.8)

$$\begin{aligned} \mathfrak{D}'_n &= X_{n+1} \mathfrak{D}_{n+1} = X_n \mathfrak{D}_{n+1} = X_n \mathfrak{D}_{C_{n+1}} = X_n (D_{C_{n+1}} (\mathfrak{D}_n + U^n \mathfrak{L}))^- = \\ &= (D_{C'_n} \mathfrak{R}_n)^- = \mathfrak{D}_{C'_n}, \end{aligned}$$

relations (2.1) and (2.2) are satisfied by  $\{C'_n\}_{1 \leq n < v}$  (of course with  $v$ ,  $\mathfrak{D}_n$ ,  $\mathfrak{L}$  and  $\mathfrak{L}'$  replaced by  $v-1$ ,  $\mathfrak{D}'_n$ ,  $U\mathfrak{L}$  and  $U' \mathfrak{L}'$ ). Also, by virtue of (2.10)<sub>1</sub> and (1.10), we have

$$\begin{aligned} (2.13)_1 \quad C'_1 (D_{A_1} (h + l_1) + Ul_2) &= C_2 X_1^{-1} (D_{A_1} (h + l_1) + Ul_2) = \\ &= C_2 (X_1^{-1} D_{A_1} (h + l_1) + Ul_2) = C_2 (D_{C_1} (D_A h + l_1) + Ul_2) \quad (h \in \mathfrak{H}, l_1, l_2 \in \mathfrak{L}), \end{aligned}$$

$$\begin{aligned} (2.13)_n \quad C'_n (D_{C'_{n-1}} (\dots D_{C'_1} (D_{A_1} (h + l_1) + Ul_2) + U^2 l_3) \dots) + U^n l_{n+1} &= \\ &= C_{n+1} X_n^{-1} (D_{C'_{n-1}} (\dots) + U^n l_{n+1}) = C_{n+1} (X_n^{-1} D_{C'_{n-1}} (\dots) + U^n l_{n+1}) = \\ &= C_{n+1} (X_n^{-1} X_{n-1}^{-1} D_{C_n} X_{n-1}^{-1} (\dots) + U^n l_{n+1}) = C_{n+1} (D_{C_n} X_{n-1}^{-1} (\dots) + U^n l_{n+1}) = \dots = \\ &= C_{n+1} (D_{C_n} (\dots X_2^{-1} (D_{C'_1} (D_{A_1} (h + l_1) + Ul_2) + U^2 l_3) \dots) + U^n l_{n+1}) = \\ &= C_{n+1} (D_{C_n} (\dots (X_2^{-1} X_1^{-1} D_{C_2} X_1^{-1} (D_{A_1} (h + l_1) + Ul_2) + U^2 l_3) \dots) + U^n l_{n+1}) = \\ &= C_{n+1} (D_{C_n} (\dots (D_{C_2} X_1^{-1} (D_{A_1} (h + l_1) + Ul_2) + U^2 l_3) \dots) + U^n l_{n+1}) = \\ &= C_{n+1} (D_{C_n} (\dots (D_{C_2} (X_1^{-1} D_{A_1} (h + l_1) + Ul_2) + U^2 l_3) \dots) + U^n l_{n+1}) = \\ &= C_{n+1} (D_{C_n} (\dots (D_{C_2} (D_{C_1} (D_A h + l_1) + Ul_2) + U^2 l_3) \dots) + U^n l_{n+1}) \quad (h \in \mathfrak{H}, l_1, \dots, l_{n+1} \in \mathfrak{L}), \end{aligned}$$

where we used in order the relations (2.10)<sub>n</sub>, (2.12)<sub>n-1</sub>, (2.11)<sub>n</sub>, (2.10)<sub>n-1</sub>, ..., (2.10)<sub>2</sub>, (2.12)<sub>1</sub>, (2.11)<sub>2</sub>, (2.10)<sub>1</sub>, (2.11)<sub>1</sub> and (1.10). Now, from (2.13)<sub>1</sub>, (2.3)<sub>2</sub> and (1.8) we infer

$$\begin{aligned} C'_1 (D_{A_1} T_1 (h + l) + (U - T_1) (h + l)) &= C'_1 (D_{A_1} (Th + (U - T)h) + Ul) = \\ &= C_2 (D_{C_1} (D_A Th + (U - T)h) + Ul) = U' C_1 (D_A h + l) = U' (P'_1 - P) A_1 (h + l) = \\ &= (U' - T'_1) A_1 (h + l) \quad (h \in \mathfrak{H}, l \in \mathfrak{L}), \end{aligned}$$

thus  $C'_1$  satisfies (2.3)<sub>1</sub> (with  $\mathfrak{F}_A$  and  $C_A$  replaced by  $\mathfrak{F}_{A_1}$  and  $C_{A_1}$ ). Also, (in case  $v > 3$ ) from (2.13)<sub>2</sub>, (2.3)<sub>3</sub> and (2.13)<sub>1</sub> we infer

$$\begin{aligned} C'_2 (D_{C'_1} (D_{A_1} T_1 (h + l) + (U - T_1) (h + l)) + U^2 l_1) &= \\ &= C'_2 (D_{C'_1} (D_{A_1} (Th + (U - T)h) + Ul) + U^2 l_1) = \\ &= C_3 (D_{C_2} (D_{C_1} (D_A Th + (U - T)h) + Ul) + U^2 l_1) = \\ &= U' C_2 (D_{C_1} (D_A h + l) + Ul_1) = U' C'_1 (D_{A_1} (h + l) + Ul_1) \quad (h \in \mathfrak{H}, l, l_1 \in \mathfrak{L}), \end{aligned}$$



thus  $C'_1, C'_2$  satisfy  $(2.3)_2$  (with  $A$  and  $\mathfrak{Q}, \mathfrak{Q}'$  replaced by  $A_1$  and  $U\mathfrak{Q}, U'\mathfrak{Q}'$  respectively). Finally, in a similar way, one verifies that (in case  $v > 4$ ), by virtue of  $(2.13)_n$ ,  $(2.3)_{n+1}$  and  $(2.13)_{n-1}$ , the string  $\{C'_n\}_{1 \leq n < v-1}$  satisfies the relations  $(2.3)_n$  for all  $n$ ,  $3 \leq n < v-1$  (again with  $A, \mathfrak{Q}, \mathfrak{Q}'$  replaced by  $A_1, U\mathfrak{Q}, U'\mathfrak{Q}'$ ). This finishes the proof of the lemma.

**Lemma 2.3.** *Let  $A_1$  be an 1-PCID of  $A$ . Any  $A_1$ -cascade string (or sequence)  $\{C'_n\}_{1 \leq n < v-1}$  is the reduced string of a uniquely determined  $A$ -cascade string (or sequence) of contractions  $\{C_n\}_{1 \leq n < v}$ .*

**Proof.** If the string (or sequence)  $\{C'_n\}_{1 \leq n < v-1}$  is the reduced string (or sequence) of  $\{C_n\}_{1 \leq n < v}$ , then the last one must be defined in the following manner. Firstly,

$$(2.14)_1 \quad C_1 = C(A_1), \quad X_1|_{\mathfrak{D}_{C_1}} = X(A_1), \quad X_1|_{U\mathfrak{Q}} = I_{U\mathfrak{Q}}$$

(thus  $\mathfrak{D}_{C_1} \subset \mathfrak{H}_1$  and  $X_1$  is a unitary operator from  $\mathfrak{D}_{C_1} + U\mathfrak{Q}$  to  $\mathfrak{H}_1 = \mathfrak{D}_{A_1} + U\mathfrak{Q}$ ); then

$$(2.14)_n \quad C_n = C'_{n-1}X_{n-1}, \quad X_n(d_n + U^n l) = X_{n-1}d_n + U^n l \quad (d_n \in \mathfrak{D}_{C_n}, l \in \mathfrak{Q})$$

( $2 \leq n < v$ ), where  $X_n$  is viewed as an operator from  $\mathfrak{D}_{C_n} + U^n \mathfrak{Q}$  to  $\mathfrak{H}_n = (X_{n-1}\mathfrak{D}_{C_n})^\perp + U^n \mathfrak{Q}$ . These definitions are consistent if they imply recurrently  $X_{n-1}\mathfrak{D}_{C_n} \subset \mathfrak{D}_{C'_{n-1}}$  ( $2 \leq n < v$ ).

However, we shall prove by induction even more, namely that

$$(2.15)_n \quad X_{n-1}\mathfrak{D}_{C_n} = \mathfrak{D}_{C'_{n-1}},$$

$$(2.16)_n \quad \mathfrak{D}_{C_n} \subset \mathfrak{H}_n$$

and that  $X_n$  is unitary (for  $2 \leq n < v$ ) (for  $n=2$ , the last two statements are, by virtue of  $(2.16)_1$ , obviously true). We start by noticing that if for some,  $n$ ,  $2 \leq n < v$ , the first relation  $(2.14)_n$  makes sense and if  $X_{n-1}$  is unitary then, by the same argument as in the proof of Lemma 2.2, we infer the validity of  $(2.12)_{n-1}$ , whence that of  $(2.15)_n$ . Thus, by virtue of  $(2.14)_1$ ,  $(2.15)_2$  is also valid, so that we have completed the first induction step. In case  $v > 3$  we can therefore assume that,  $n$  being fixed,  $2 \leq n < v$ , the statements are always valid for  $n-1$ . Then, by virtue of  $(2.15)_{n-1}$  and the fact that  $\{C'_n\}_{1 \leq n < v-1}$  is an  $A_1$ -cascade, the first relation  $(2.14)_n$  makes sense; thus, by virtue of the above discussion on  $(2.15)_n$ , we infer that this relation is valid. Therefore, using once again the fact that  $\{C'_n\}_{1 \leq n < v-1}$  is an  $A_1$ -cascade, from the second relation  $(2.14)_n$  we obtain that  $X_n$  is unitary, while from the second relation  $(2.14)_{n-1}$  and  $(2.16)_{n-1}$  we obtain  $(2.16)_n$ . Thus the  $n^{\text{th}}$  inductive step is completed and consequently the string (or sequence)  $\{C_n\}_{1 \leq n < v}$  is consistently defined. By this very definition, it is plain that  $\{C_n\}_{1 \leq n < v}$  satisfies conditions  $(2.1)$ ,  $(2.2)$  and  $(2.3)_1$ . Now we can establish, as in the proof of Lemma 2.2, the relations  $(2.13)_n$  ( $1 \leq n < v-1$ ) and subsequently infer the relations  $(2.3)_n$  ( $2 \leq n < v$ ) for  $\{C_n\}_{1 \leq n < v}$  from

the fact that  $\{C'_n\}_{1 \leq n < v}$ , being an  $A_1$ -cascade, satisfies (2.3)<sub>n</sub> ( $1 \leq n < v-1$ ; of course with  $A$ ,  $\mathfrak{Q}$  and  $\mathfrak{Q}'$  replaced by  $A_1$ ,  $U'$  and  $U'$ , respectively). In this manner we conclude that  $\{C_n\}_{1 \leq n < v}$  is an  $A$ -cascade. Actually, the proof of the lemma is now completed.

We can, and shall, now define a mapping from  $A$ -cascade strings to PCID's of  $A$ , for any contractions  $T, T'$  and  $A \in \mathcal{J}(T', T)$ . Namely, for given  $T, T', A$ ,  $N=1, 2, \dots$ , and  $A$ -cascade string  $\{C_n\}_1^N$  (of length  $N$ ) we shall define an  $N$ -PCID  $A_N(A; C_1, \dots, C_N)$  by the following recurrent formula

$$(2.17)_1 \quad A_1(A; C_1) = A_1$$

where  $A_1$  is the 1-PCID (yielded by Lemma 1.2) such that  $C(A_1) = C_1$ , and

$$(2.17)_N \quad A_N(A; C_1, \dots, C_N) = A_{N-1}(A_1; C'_1, \dots, C'_{N-1})$$

where  $\{C'_n\}_1^{N-1}$  is the reduced string of  $\{C_n\}_1^N$  ( $N=2, 3, \dots$ ). (Actually, one should write  $A_N(A; T', T; U', U; C_1, \dots, C_N)$  instead of  $A_N(A; C_1, \dots, C_N)$  since this operator depends also on  $T, T'$  and the concrete constructions of the isometric dilations  $U, U'$  of  $T, T'$ , respectively; thus (2.17)<sub>N</sub> should be written in the form

$$A_N(A; T', T; U', U; C_1, \dots, C_N) = A_{N-1}(A_1; T'_1, T_1; U'_1, U_1; C'_1, \dots, C'_{N-1}).$$

However, when no confusion seems possible, we shall not complicate the notations with this preciseness.)

The consistence of the definitions (2.17)<sub>N</sub> ( $N=2, 3, \dots$ ) is a direct consequence of Lemma 2.2 and the fact that any  $(N-1)$ -PCID of a 1-PCID of  $A$  is an  $N$ -PCID of  $A$ . Also by an obvious inductive argument it follows that

$$(2.18)_N \quad P'_{N-1} A_N(A; C_1, \dots, C_N) = A_{N-1}(A; C_1, \dots, C_{N-1})(P_{N-1}|H_N)$$

( $N=2, 3, \dots$ ), i.e.  $A_N(A; C_1, \dots, C_N)$  is a 1-PCID of  $A_{N-1}(A; C_1, \dots, C_{N-1})$ .

**Proposition 2.1.** For  $N=1, 2, \dots$ , and  $T, T', A \in \mathcal{J}(T', T)$  fixed, the mapping

$$(2.19)_N \quad \{C_n\}_1^N \rightarrow A_N(A; C_1, \dots, C_N)$$

establishes a one-to-one correspondence between the  $A$ -cascade strings (of length  $N$ ) and the  $N$ -PCID's of  $A$ .

**Proof.** For  $N=1$ , the statement in the proposition reduces to the first statement in Lemma 1.2. Therefore we assume that the statement is also true for  $N=m-1 \geq 1$ . Let moreover  $A_m$  be an  $m$ -PCID of  $A$  and let  $A_1$  be the 1-PCID of  $A$  defined by (1.2)<sub>1</sub> (with  $v=m$ ). Then,  $A_m$  is an  $(m-1)$ -PCID of  $A_1$ , thus by the inductive assumption, there exists a uniquely determined  $A_1$ -cascade string  $\{C'_n\}_1^{m-1}$  such that

$$(2.20)_m \quad A_m = A_{m-1}(A_1; C'_1, \dots, C'_{m-1}).$$

By virtue of Lemma 2.3 there exists a unique  $A$ -cascade string  $\{C_n\}_1^N$  such that  $\{C'_n\}_1^{m-1}$  is the  $(A_1$ -cascade) reduced string of  $\{C_n\}_1^m$ ; moreover (see (2.14)<sub>1</sub>)  $C_1 = C(A_1)$ . Therefore, from (2.20)<sub>m</sub> and (2.17)<sub>m</sub> we infer that  $A_m$  is of the form  $A_m = A_m(A; C_1, C_2, \dots, C_m)$ , where  $\{C_n\}_1^m$  is the above (uniquely determined)  $A$ -cascade string. This finishes the proof of the proposition.

**Lemma 2.4.** *Within the frame of Proposition 2.1 we have*

$$(2.21)_N \quad \|D_{A_N} h\| = \|D_{C_N} D_{C_{N-1}} \dots D_{C_1} D_A h\| \quad (h \in \mathfrak{H}),$$

where  $A_N = A_N(A; C_1, C_2, \dots, C_N)$  and  $\{C_n\}_1^N$  is an  $A$ -cascade string.

**Proof.** Relation (2.21)<sub>1</sub> follows directly from Lemma 1.2.

Suppose relation (2.21)<sub>N</sub> be true for  $N = m-1 \geq 0$ . Then from (2.17)<sub>m</sub> we infer

$$(2.22) \quad \|D_{A_m} h\| = \|D_{A'_{m-1}} h\| = \|D_{C'_{m-1}} \dots D_{C'_1} D_{A_1} h\| \quad (h \in \mathfrak{H})$$

where  $A'_{m-1} = A_{m-1}(A; C'_1, \dots, C'_{m-1})$ ,  $A_1$  is the 1-PCID of  $A$  defined by (1.2)<sub>1</sub> (with  $v=m$ ) and  $\{C'_n\}_1^{m-1}$  is the reduced string of  $\{C_n\}_1^m$ . But, by virtue of Lemma 1.2 we have  $D_{A_1} h = X(A_1) D_{C_1} D_A h$  so that, if  $\{X_n\}_1^N$  is the  $\{C_n\}_1^N$ -extension of  $X(A_1)$ , we obtain

$$\begin{aligned} D_{C'_{N-1}} \dots D_{C'_1} D_{A_1} h &= D_{C'_{N-1}} \dots D_{C'_1} X_1 D_{C_1} D_A h = D_{C'_{N-1}} \dots D_{C'_1} X_1 D_{C_2} D_{C_1} D_A h = \\ &= D_{C'_{N-1}} \dots D_{C'_2} X_2 D_{C_2} D_{C_1} D_A h = \dots = D_{C'_{N-1}} X_{N-2} D_{C_{N-1}} \dots D_{C_1} D_A h = \\ &= X_{N-1} D_{C_N} D_{C_{N-1}} \dots D_{C_1} D_A h \quad (h \in \mathfrak{H}), \end{aligned}$$

where we used, in order, relations (2.11)<sub>1</sub>, (2.12)<sub>1</sub>, (2.11)<sub>2</sub>, ..., (2.11)<sub>N-1</sub>, (2.12)<sub>N-1</sub>. Since  $X_{N-1}$  is unitary, from (2.22) it follows that (2.21)<sub>m</sub> is also valid. This completes the proof.

**Proposition 2.2.** *The mapping*

$$(2.23) \quad \{C_n\}_1^\infty \rightarrow A_\infty(A; C_1, C_2, \dots) = \text{strong lim } A_N(A; C_1, \dots, C_N) P_N$$

*establishes a one-to-one correspondence between all the  $A$ -cascade sequences and all the CID's of  $A$ . Moreover,  $A_\infty = A_\infty(A; C_1, C_2, \dots)$  is an isometry if and only if*

$$(2.24) \quad \|D_{C_N} D_{C_{N-1}} \dots D_{C_1} D_A h\| \rightarrow 0 \quad (h \in \mathfrak{H}; N \rightarrow \infty).$$

**Proof.** The first statement of the proposition follows at once from Remark 1.1, Proposition 2.1 and (2.18)<sub>N</sub> ( $N=2, 3, \dots$ ). Concerning the second, we remark that (2.24) holds if and only if

$$(2.25) \quad \|D_{A_N} h\| \rightarrow 0 \quad (h \in \mathfrak{H}; N \rightarrow \infty),$$

where  $A_N = A_N(A; C_1, \dots, C_N)$  ( $N=1, 2, \dots$ ).

From the first statement it follows that

$$\|D_{A_\infty} h\|^2 = \|h\|^2 - \|A_\infty h\|^2 = \|h\|^2 - \lim_{N \rightarrow \infty} \|A_N h\|^2 = \lim_{N \rightarrow \infty} \|D_{A_N} h\|^2$$

and consequently (2.25) (or equivalently (2.24)) holds if and only if  $D_{A_\infty} \mathfrak{H} = O$ , that is, if  $A_\infty|_{\mathfrak{H}}$  is isometric. Thus it remains only to prove that the last property implies that  $A_\infty$  is isometric. Or, since  $A_\infty U = U' A_\infty$ , it follows at once that  $A_\infty|_{U^n \mathfrak{H}}$  is isometric for  $n=0, 1, 2, \dots$ ; in its turn, this implies that  $D_{A_\infty}|_{U^n \mathfrak{H}} = O$  ( $n=0, 1, 2, \dots$ ).

Since the spaces  $U^n \mathfrak{H}$  ( $n=0, 1, 2, \dots$ ) span  $\mathfrak{R}$ , we conclude that  $D_{A_\infty} = O$ , i.e.  $A_\infty$  is isometric.

3. Propositions 2.1 and 2.2 reduce the study of all PCID's and CID's of an  $A \in \mathcal{J}(T', T)$  (where  $T, T'$  and  $A$  are some given contractions) to that of  $A$ -cascade strings and sequences. However an  $A$ -cascade string or sequence is a rather involved concept. Therefore we shall show that the study can be actually confined to more transparent concepts, one of which is defined in the following.

**Definition 3.1.** A string (or sequence)  $\{\Gamma_n\}_{1 \leq n < \infty}$  of operators will be called an  $A$ -choice string (or sequence) if each  $\Gamma_n$  ( $1 \leq n < \infty$ ) is a contraction acting from  $\mathfrak{R}_A$  to  $\mathfrak{R}'_A$  (if  $n=1$ ) and from  $\mathfrak{D}_{\Gamma_{n-1}}$  to  $\mathfrak{D}_{\Gamma_{n-1}}^*$  (if  $n \geq 2$ ). (Thus if  $\{\Gamma_n\}_1^N$  is an  $A$ -choice string, then for any contraction  $\Gamma_{N+1} \in L(\mathfrak{D}_{\Gamma_N}, \mathfrak{D}_{\Gamma_N}^*)$ ,  $\{\Gamma_n\}_1^{N+1}$  is also an  $A$ -choice string; this is the justification of the terminology.)

In this section we shall establish a natural connection between the  $A$ -cascade strings (or sequences) and the  $A$ -choice strings (or sequences). To this aim we need some simple facts, rather known, which, for the sake of completeness, will be collected in the following:

**Lemma 3.1.**  $\mathfrak{G}, \mathfrak{G}'$  and  $\mathfrak{G}_0$  be some Hilbert spaces,  $\mathfrak{G}_0$  being a subspace of  $\mathfrak{G}$ , and let  $C_0: \mathfrak{G}_0 \rightarrow \mathfrak{G}'$  be a contraction. Then the formulas

$$(3.1) \quad D_{C_0^*} \Gamma(C_0, C) = C|_{\mathfrak{G} \ominus \mathfrak{G}_0}, \quad C(C_0, \Gamma) = C_0 Q + D_{C_0^*} \Gamma(I - Q)$$

(where  $Q$  denotes the orthogonal projection of  $\mathfrak{G}$  onto  $\mathfrak{G}_0$ ), establishes a one-to-one correspondence between all the contractions  $C: \mathfrak{G} \rightarrow \mathfrak{G}'$  such that

$$(3.2) \quad C|_{\mathfrak{G}_0} = C_0,$$

and all the contractions  $\Gamma: \mathfrak{G} \ominus \mathfrak{G}_0 \rightarrow \mathfrak{D}_{C_0^*}$ . Moreover, the formulas

$$(3.3) \quad Z D_\Gamma = R D_C|_{\mathfrak{G} \ominus \mathfrak{G}_0}, \quad Z_* D_{\Gamma^*} D_{C_0^*} = D_{C_*}, \quad Z' D_{C_0} = D_C|_{\mathfrak{G}_0}$$

(where  $R$  denotes the orthogonal projection of  $\mathfrak{D}_C$  onto  $\mathfrak{D}_C \ominus (D_C \mathfrak{G}_0)^\perp$  - define unitary operators  $Z = Z(C_0, C)$  from  $\mathfrak{D}_\Gamma$  to  $\mathfrak{D}_C \ominus (D_C \mathfrak{G}_0)^\perp$ ,  $Z_* = Z_*(C_0, C)$  from  $\mathfrak{D}_{\Gamma^*}$  to  $\mathfrak{D}_{C_*}$  and  $Z' = Z'(C_0, C)$  from  $\mathfrak{D}_{C_0}$  to  $(D_C \mathfrak{G}_0)^\perp$ ; also

$$(3.4) \quad Z \mathfrak{D}_\Gamma = \mathfrak{D}_C \ominus Z' \mathfrak{D}_{C_0}.$$

Proof. Let  $C: \mathfrak{G} \rightarrow \mathfrak{G}'$  be a contraction with the property. (3.2). Then,

$$(3.5) \quad \|QC^*g'\|^2 + \|(I-Q)C^*g'\|^2 = \|C^*g'\|^2 \leq \|g'\|^2 \quad (g' \in \mathfrak{G}'),$$

$$(QC^*g', g_0) = (C^*g', g_0) = (g', Cg_0) = (g', C_0g_0) = (C_0^*g', g_0) \quad (g' \in \mathfrak{G}', g_0 \in \mathfrak{G}_0)$$

whence

$$(3.6) \quad QC^* = C_0^*$$

and therefore, by (3.5),

$$(3.7) \quad \|(I-Q)C^*g'\| \leq \|D_{C_0^*}g'\| \quad (g' \in \mathfrak{G}').$$

It follows that there exists a unique contraction  $\Gamma^*: \mathfrak{D}_{C_0^*} \rightarrow \mathfrak{G} \ominus \mathfrak{G}_0$  such that

$$(3.8) \quad \Gamma^*D_{C_0^*} = (I-Q)C^*.$$

Consequently, setting  $\Gamma(C_0, C) = \Gamma \in L(\mathfrak{G} \ominus \mathfrak{G}_0, \mathfrak{D}_{C_0^*})$  we obtain the first relation (3.1). Conversely, if we are given a contraction  $\Gamma: \mathfrak{G} \ominus \mathfrak{G}_0 \rightarrow \mathfrak{D}_{C_0^*}$  and if we define  $C = C(C_0, \Gamma)$  by the second relation (3.1), then (3.2) and (3.8) are plainly satisfied; consequently we obtain (3.6) with the same argument as above. It follows

$$(3.9) \quad \|C^*g'\|^2 = \|QC^*g'\|^2 + \|(I-Q)C^*g'\|^2 = \|C_0^*g'\|^2 + \|\Gamma^*D_{C_0^*}g'\|^2 \leq \\ \leq \|C_0^*g'\|^2 + \|D_{C_0^*}g'\|^2 = \|g'\|^2 \quad (g' \in \mathfrak{G}')$$

hence  $C$  is a contraction; finally, (3.8) shows that  $\Gamma(C_0, C) = \Gamma$ .

This completes the proof of the first statement in the lemma. The statements on  $Z_*$  and  $Z'$  follow readily from (3.9) and (3.2), respectively. Concerning the statement on  $Z$ , we note that

$$\begin{aligned} \|D_Cg\|^2 &= \|g\|^2 - \|Cg\|^2 = \|(I-Q)g\|^2 + \|Qg\|^2 - \|C_0Qg + D_{C_0^*}\Gamma(I-Q)g\|^2 = \\ &= \|(I-Q)g\|^2 + \|Qg\|^2 - \|C_0Qg\|^2 - 2\operatorname{Re}(C_0Qg, D_{C_0^*}\Gamma(I-Q)g) - \|D_{C_0^*}\Gamma(I-Q)g\|^2 = \\ &= \|(I-Q)g\|^2 + \|D_{C_0}Qg\|^2 - 2\operatorname{Re}(D_{C_0^*}C_0Qg, \Gamma(I-Q)g) - \|D_{C_0^*}\Gamma(I-Q)g\|^2 = \\ &= \|(I-Q)g\|^2 + \|D_{C_0}Qg\|^2 - 2\operatorname{Re}(C_0D_{C_0}Qg, \Gamma(I-Q)g) - \|D_{C_0^*}\Gamma(I-Q)g\|^2 = \\ &= \|D_{\Gamma}(I-Q)g\|^2 + \|C_0^*\Gamma(I-Q)g\|^2 + \|D_{C_0}Qg\|^2 - 2\operatorname{Re}(D_{C_0}Qg, C_0^*\Gamma(I-Q)g) = \\ &= \|D_{\Gamma}(I-Q)g\|^2 + \|D_{C_0}Qg - C_0^*\Gamma(I-Q)g\|^2 \quad (g \in \mathfrak{G}) \end{aligned}$$

whence

$$(3.10) \quad \|D_Cg + D_Cg_0\|^2 = \|D_{\Gamma}g\|^2 + \|D_{C_0}g_0 - C_0^*\Gamma g\|^2 \quad (g \in \mathfrak{G} \ominus \mathfrak{G}_0, g_0 \in \mathfrak{G}_0).$$

But since  $C_0^*\Gamma(\mathfrak{G} \ominus \mathfrak{G}_0) \subset C_0^*\mathfrak{D}_{C_0^*} \subset \mathfrak{D}_{C_0}$ , from relation (3.10) it follows that

$$\|RD_Cg\|^2 = \inf_{g_0 \in \mathfrak{G}_0} \|D_Cg + D_Cg_0\|^2 = \|D_{\Gamma}g\|^2 \quad (g \in \mathfrak{G} \ominus \mathfrak{G}_0).$$

This shows that the definition of  $Z$  is meaningful and that  $Z$  is unitary. Since (3.4) is now obvious, the proof is completed.

We now return to the aim of this section, stated before Lemma 3.1, by considering an  $A$ -cascade string  $\{C_n\}_1^N$  (where  $T, T'$  and  $A$  are as in Sec. 2). We set

$$(3.11) \quad \mathfrak{G}_{01} = \mathfrak{F}_A, \quad \mathfrak{G}_1 = \mathfrak{D}_A + \mathfrak{L}, \quad \mathfrak{G}'_1 = \mathfrak{L}',$$

$$\mathfrak{G}_{0n} = Y_{n-1} \mathfrak{D}_{n-2} + U^{n-1} \mathfrak{L}, \quad \mathfrak{G}_n = \mathfrak{D}_{n-1} + U^{n-1} \mathfrak{L}, \quad \mathfrak{G}'_n = U'^{n-1} \mathfrak{L}' \quad (n > 1)$$

and we define the contractions  $C_{0n}: \mathfrak{G}_{0n} \rightarrow \mathfrak{G}'_n$  ( $n=1, 2, \dots, N$ ) by

$$(3.12) \quad C_{01} = C_A, \quad C_{0n} Y_{n-1} | \mathfrak{D}_{n-2} =$$

$$= U' C_{n-1} | \mathfrak{D}_{n-2}, \quad C_{0n} U | U^{n-2} \mathfrak{L} = U' C_{n-1} | U^{n-2} \mathfrak{L} \quad (n > 1).$$

By virtue of (2.3)<sub>1</sub> and (2.6)<sub>n</sub> (for  $n > 1$ ) we have

$$(3.13) \quad C_n | \mathfrak{G}_{0n} = C_{0n} \quad (n = 1, 2, \dots, N).$$

Therefore, Lemma 3.1 yields the operators

$$(3.14)_n \quad \Gamma'_n = \Gamma(C_{0n}, C_n), \quad Z_n = Z(C_{0n}, C_n), \quad Z_{*n} = Z_*(C_{0n}, C_n), \quad Z'_n = Z'(C_{0n}, C_n)$$

for  $n=1, 2, \dots, N$ . (In the sequel, when a more precise notation will seem necessary, we shall write  $\Gamma'_n = \Gamma'_n(C_1, \dots, C_n)$ ,  $Z_n = Z_n(C_1, \dots, C_n)$ , ... instead of  $\Gamma'_n, Z_n, \dots$ ).

Lemma 3.2. For  $2 \leq n \leq N$  the range of  $Z'_{n-1}$  is  $Y_{n-1} \mathfrak{D}_{n-2}$  and  $\Gamma'_n$  is a contraction from  $Z_{n-1} \mathfrak{D}_{\Gamma'_{n-1}}$  to  $U' \mathfrak{D}_{C_{n-1}^*}$ .

Proof. For proving

$$(3.15)_n \quad Z'_{n-1} \mathfrak{D}_{C_{0,n-1}} = Y_{n-1} \mathfrak{D}_{n-2}$$

for  $n=2, \dots, N$ , we note firstly that (3.15)<sub>2</sub> follows from (3.3) and (2.5)<sub>1</sub>, by the relations

$$Z'_1 D_{C_A} (D_A T h + (U - T) h) = D_{C_1} (D_A T h + (U - T) h) = Y_1 D_A h \quad (h \in \mathfrak{H}).$$

For  $n > 2$  we have, by virtue of (3.3), (3.12) and (2.5)<sub>n-1</sub>,

$$Z'_{n-1} D_{C_{0,n-1}} (Y_{n-2} d + U^{n-2} l) = D_{C_{n-1}} (Y_{n-2} d + U^{n-2} l) =$$

$$= Y_{n-1} D_{C_{n-2}} (d + U^{n-3} l) \quad (d \in \mathfrak{D}_{n-3}, l \in \mathfrak{L}),$$

from which (3.15)<sub>n</sub> follows at once. Concerning the second statement in the lemma, we first notice that Lemma 3.1 yields

$$\Gamma'_n \in L(\mathfrak{G}_n \ominus \mathfrak{G}_{0n}, \mathfrak{D}_{C_{0n}^*}).$$

But, by virtue of (3.11), (3.15)<sub>n</sub> and (3.4) we have firstly

$$(3.16)_n \quad \mathfrak{G}_n \ominus \mathfrak{G}_{0n} = \mathfrak{D}_{n-1} \ominus Y_{n-1} \mathfrak{D}_{n-2} = \mathfrak{D}_{C_{n-1}} \ominus Z'_{n-1} \mathfrak{D}_{C_{0,n-1}} = Z_{n-1} \mathfrak{D}_{\Gamma'_{n-1}},$$

while, by virtue of (3.12), we have  $C_{0n} C_{0n}^* = U' C_{n-1} C_{n-1}^* U'^* | U'^{n-1} \mathfrak{L}'$  whence

$$D_{C_{0n}^*}^2 = U' D_{C_{n-1}^*}^2 U'^* | U'^{n-1} \mathfrak{L}', \quad D_{C_{0n}^*} = U' D_{C_{n-1}^*} U'^* | U'^{n-1} \mathfrak{L}'$$

and hence

$$(3.17)_n \quad \mathfrak{D}_{C_{0n}^*} = U' \mathfrak{D}_{C_{n-1}^*}.$$

We can thus conclude that  $\Gamma'_n \in L(Z_{n-1} \mathfrak{D}_{\Gamma'_{n-1}}, U' \mathfrak{D}_{C_{n-1}^*})$ , completing the proof of the lemma.

We shall associate with our  $A$ -cascade string  $\{C_n\}_1^N$  an  $A$ -choice string  $\{\Gamma_n\}_1^N$  in the following manner. We set

$$(3.18)_1 \quad \Gamma_1 = W_A \Gamma'_1,$$

and

$$(3.19)_1 \quad W_1 = I_{\mathfrak{D}_{\Gamma_1}}, \quad W_{*1} = W_A^* | \mathfrak{D}_{\Gamma_1^*}.$$

Since  $W_A \in \mathfrak{L}(\mathfrak{D}_{C_A^*}, \mathfrak{R}_A')$  is unitary (see Lemma 1.1), we have obviously  $D_{\Gamma_1} = D_{\Gamma'_1}$ ,  $W_A D_{\Gamma_1^*} = D_{\Gamma_1^*} W_A$  hence the operators

$$(3.20)_1 \quad W_1: \mathfrak{D}_{\Gamma_1} \rightarrow \mathfrak{D}_{\Gamma'_1}, \quad W_{*1}: \mathfrak{D}_{\Gamma_1^*} \rightarrow \mathfrak{D}_{\Gamma_1'^*},$$

defined by formula (3.19)<sub>1</sub>, are unitary; moreover, we also have

$$(3.21)_1 \quad \Gamma_1 \in L(\mathfrak{R}_A, \mathfrak{R}_A'),$$

thus  $\{\Gamma_1\}$  is an  $A$ -choice string (of length 1); this will be associated with our  $A$ -cascade string if  $N=1$ . If  $N>1$ , we appeal to the following

**Lemma 3.3.** *Let  $N>1$ . Then formulas (3.18)<sub>1</sub>, (3.19)<sub>1</sub>, and, for  $2 \leq n \leq N$ ,*

$$(3.18)_n \quad \Gamma_n = W_{*n-1}^* Z_{*n-1}^* U'^* \Gamma'_n Z_{n-1} W_{n-1},$$

$$(3.19)_n \quad W_n = Z_{n-1} W_{n-1} | \mathfrak{D}_{\Gamma_n}, \quad W_{*n} = U' Z_{*n-1} W_{*n-1} | \mathfrak{D}_{\Gamma_n^*}$$

define an  $A$ -choice string  $\{\Gamma_n\}_1^N$  and unitary operators

$$(3.20)_n \quad W_n: \mathfrak{D}_{\Gamma_n} \rightarrow \mathfrak{D}_{\Gamma'_n}, \quad W_{*n}: \mathfrak{D}_{\Gamma_n^*} \rightarrow \mathfrak{D}_{\Gamma_n'^*} \quad (1 \leq n \leq N).$$

**Proof.** Proceeding by recurrence, we notice that the statements concerning  $\Gamma_1$ ,  $W_1$  and  $W_{*1}$  were already established above. Assuming that those concerning  $W_{m-1}$  and  $W_{*m-1}$  (where  $m-1 \geq 1$ ,  $m \leq M$ ) are also established we infer by virtue of Lemma 3.2 and (3.20)<sub>m-1</sub> that the relation

$$(3.21)_n \quad \Gamma_n \in L(\mathfrak{D}_{\Gamma_{n-1}}, \mathfrak{D}_{\Gamma_{n-1}^*})$$

is valid for  $n=m$ . From this we obtain

$$Z_{m-1} W_{m-1} \Gamma_m^* \Gamma_m = \Gamma_m^* \Gamma_m Z_{m-1} W_{m-1},$$

$$U' Z_{*m-1} W_{*m-1} \Gamma_m \Gamma_m^* = \Gamma_m^* \Gamma_m^* U' Z_{*m-1} W_{*m-1},$$

whence

$$Z_{m-1} W_{m-1} D_{\Gamma_m} = D_{\Gamma'_m} Z_{m-1} W_{m-1}, \quad U' Z_{*m-1} W_{*m-1} D_{\Gamma_m^*} = D_{\Gamma_m'^*} U' Z_{*m-1} W_{*m-1}.$$

From these relations it follows readily that formula  $(3.19)_m$  defines the unitary operators  $(3.20)_m$ . Thus the operators  $W_n$  and  $W_{*n}$  ( $1 \leq n \leq N$ ) are unitary and  $(3.21)_n$  is true for all  $n$ ,  $1 \leq n \leq N$ , which means that  $\{\Gamma_n\}_1^N$  is an  $A$ -choice sequence. This finishes the proof.

It is plain that the operators  $\Gamma_n$ ,  $W_n$  and  $W_{*n}$  ( $1 \leq n \leq N$ ) yielded by the preceding argument depend only on  $C_1, C_2, \dots, C_n$  (and of course on  $A, T, T'$  and  $U, U'$ ). Therefore we shall denote them by  $\Gamma_n(C_1, \dots, C_n)$ ,  $W_n(C_1, \dots, C_n)$  and  $W_{*n}(C_1, \dots, C_n)$ . (When a confusion seems possible we shall explicitate also the dependence on  $A, T, T', U, U'$ , for instance  $\Gamma_n(A; T, T'; U, U'; C_1, \dots, C_n)$  for  $\Gamma_n$  etc.)

**Proposition 3.1.** *For  $v=2, 3, \dots, \infty$  and  $T, T', A \in \mathcal{J}(T', T)$ , fixed, the mapping*

$$(3.22)_v \quad \{C_n\}_{1 \leq n < v} \rightarrow \{\Gamma_n(C_1, \dots, C_n)\}_{1 \leq n < v}$$

*establishes a one-to-one correspondence between all the  $A$ -cascade strings (if  $v < \infty$ ), respectively sequences (if  $v = \infty$ ) and all the  $A$ -choice strings (of length  $v-1$ , if  $v < \infty$ ), respectively sequences (if  $v = \infty$ ).*

**Proof.** The case  $v = \infty$  follows immediately from the case  $v < \infty$ .

Since the case  $v=2$  is a direct consequence of Lemma 3.1, we shall assume now that the proposition is valid if  $v=m \geq 2$ .

Let  $\{\Gamma_n\}_{1 \leq n < m+1}$  be any  $A$ -choice string. Then by our assumption there exists a unique  $A$ -cascade string  $\{C_n\}_{1 \leq n < m}$  such that

$$(3.23)_n \quad \Gamma_n = \Gamma_n(C_1, C_2, \dots, C_m) \quad (1 \leq n < m).$$

Therefore, by virtue of Lemmas 3.1 and 3.2, the operators

$$\Gamma' = \Gamma'_{m-1}(C_1, \dots, C_{m-1}): \mathfrak{G}_{m-1} \ominus \mathfrak{G}_{0, m-1} \rightarrow \mathfrak{D}_{C_{0, m-1}}^*$$

and

$$Z = Z_{m-1} = Z(C_1, \dots, C_{m-1}): \mathfrak{D}_{\Gamma'_{m-1}} \rightarrow \mathfrak{G}_m \ominus \mathfrak{G}_{0m} \quad (\text{see } (3.16)_m)$$

$$Z_* = Z_{*m-1} = Z_*(C_1, \dots, C_{m-1}): \mathfrak{D}_{\Gamma'^*_{m-1}} \rightarrow \mathfrak{D}_{C_{*m-1}}^*,$$

$$W = W_{m-1} = W(C_1, \dots, C_{m-1}): \mathfrak{D}_{\Gamma_{m-1}} \rightarrow \mathfrak{D}_{\Gamma'_{m-1}},$$

$$W_* = W_{*m-1} = W_*(C_1, \dots, C_{m-1}): \mathfrak{D}_{\Gamma^*_{m-1}} \rightarrow \mathfrak{D}_{\Gamma'^*_{m-1}}$$

are also uniquely determined, and  $\Gamma'$  is a contraction while  $Z, Z_*, W, W_*$  are unitary. Setting

$$(3.24) \quad \Gamma' = U' Z_* W_* \Gamma_m W^* Z^*$$

we obtain a contraction from  $\mathfrak{G}_m \ominus \mathfrak{G}_{0m}$  to  $\mathfrak{D}_{C_{0m}}^*$  (see (3.11), (3.12) and (3.17)<sub>m</sub>). By virtue of Lemma 3.1, there exists a uniquely determined contraction  $C_m: \mathfrak{G}_m \rightarrow \mathfrak{G}'_m$  such that  $C_m = C(C_{0m}, \Gamma')$ . Comparing (3.11), (3.12) and (3.13) (in the case  $n=m$ ) with (2.6)<sub>m</sub> we see that  $\{C_n\}_{1 \leq n < m+1}$  is an  $A$ -cascade string.



Comparing (3.24) with (3.18)<sub>m</sub>, we finally see that (3.23)<sub>m</sub> is also valid. Thus we verified that the mapping (3.22)<sub>m+1</sub> is surjective. Since the last term in  $\{C_n\}_{1 \leq n < m+1}$  is necessarily of the form  $C_m = C(C_{0m}, \Gamma')$ , where  $\Gamma'$  is given by (3.24), the mapping is also injective.

Now the proposition is concluded by induction.

4. In this section we shall associate the CID's of an  $A \in \mathcal{J}(T', T)$  with  $T, T'$  and  $A$  as in the preceding sections, to a more usual concept, namely to contractive analytic functions ([16], Ch. V.). As preparation, we shall now discuss the preceding sections in a very particular case, namely that of an arbitrary contraction  $\Gamma$  from  $\mathfrak{R}$  to  $\mathfrak{R}'$  (where  $\mathfrak{R}$  and  $\mathfrak{R}'$  are two Hilbert spaces), considered as intertwining the corresponding null operators  $0_{\mathfrak{R}}, 0_{\mathfrak{R}'}$ , i.e.  $\Gamma \in \mathcal{J}(0_{\mathfrak{R}'}, 0_{\mathfrak{R}})$ .

On this purpose, for the operator  $0_{\mathfrak{R}}$  we shall choose as minimal isometric dilation  $V_{\mathfrak{R}}$  the canonical multiplication shift

$$V_{\mathfrak{R}}f(z) = zf(z) \quad (|z| < 1)$$

on  $H^2(\mathfrak{R})$ , where  $\mathfrak{R}$  is identified with the space of constant functions in  $H^2(\mathfrak{R})$ ;\*) the minimal isometric dilation  $V_{\mathfrak{R}'}$  of  $0_{\mathfrak{R}'}$  will be chosen in the obvious similar way. Since any CID of  $\Gamma$  is a contraction intertwining  $V_{\mathfrak{R}}$  and  $V_{\mathfrak{R}'}$ , it is the multiplication operator by a contractive analytic function  $\{\mathfrak{R}, \mathfrak{R}', \Gamma(z)\}$  (see [16], Ch. V, Sec. 3), which obviously must satisfy the condition  $\Gamma(0) = \Gamma$ . Since the converse fact is also obvious, we can state the following consequence of our previous results.

**Lemma 4.1.** *Let  $\Gamma: \mathfrak{R} \rightarrow \mathfrak{R}'$  be an arbitrary fixed contraction. Then Propositions 2.2 and 3.1 with  $T=0_{\mathfrak{R}}, T'=0_{\mathfrak{R}'}$ , and  $A=\Gamma \in \mathcal{J}(T'; T)$  yield a one-to-one correspondence between all contractive analytic functions  $\{\mathfrak{R}, \mathfrak{R}', \Gamma(z)\}$  such that  $\Gamma(0)=\Gamma$  and all  $\Gamma$ -choice sequences.*

**Remark 4.1.** We recall that within the frame of the preceding discussion, (1.5) and (1.5)' take the form

$$(4.1) \quad \mathfrak{F}_{\Gamma} = V_{\mathfrak{R}} \mathfrak{R}, \quad \mathfrak{R}_{\Gamma} = (\mathfrak{D}_{\Gamma} + V_{\mathfrak{R}} \mathfrak{R}) \ominus \mathfrak{F}_{\Gamma} = \mathfrak{D}_{\Gamma}$$

and

$$(4.1)' \quad \begin{cases} \mathfrak{F}_{\Gamma}' = \{D_{\Gamma} r \oplus V_{\mathfrak{R}'} \Gamma r : r \in \mathfrak{R}\}, \\ \mathfrak{R}_{\Gamma}' = (\mathfrak{D}_{\Gamma} \oplus V_{\mathfrak{R}'} \mathfrak{R}') \ominus \mathfrak{F}_{\Gamma}' = \{r \oplus V_{\mathfrak{R}'} r' : D_{\Gamma} r + \Gamma^* r' = 0, \quad r \in \mathfrak{D}_{\Gamma}, r' \in \mathfrak{R}'\}. \end{cases}$$

**Lemma 4.2.** *The formula*

$$(4.2) \quad \omega(\Gamma) r' = (-\Gamma^* r') \oplus V_{\mathfrak{R}'} D_{\Gamma^*} r' \quad (r' \in \mathfrak{D}_{\Gamma^*})$$

*defines a unitary operator from  $\mathfrak{D}_{\Gamma^*}$  to  $\mathfrak{R}_{\Gamma}'$ .*

\*) For the Hardy spaces  $H^2(\mathfrak{R})$ , where  $\mathfrak{R}$  is a Hilbert space, see [16], Ch. V.

**Proof.** It is obvious, by virtue of (4.1)' and the relation  $D_r \Gamma^* = \Gamma^* D_{r^*}$ , that (4.2) defines an isometric operator  $\omega(\Gamma)$  from  $\mathfrak{D}_{r^*}$  to  $\mathfrak{R}'_r$ . Moreover, if we are given  $r \oplus V r' \in \mathfrak{R}'_r$ , then setting  $r'_1 = D_{r^*} r' - \Gamma r$  we obtain  $r'_1 \in \mathfrak{D}_{r^*}$  and

$$\begin{aligned} \omega(\Gamma) r'_1 &= (-\Gamma^* D_{r^*} r' + \Gamma^* \Gamma r) \oplus V_{\mathfrak{R}'}(D_{r^*}^2 r' - D_{r^*} \Gamma r) = \\ &= (r - D_r(\Gamma^* r' + D_r r)) \oplus V_{\mathfrak{R}'}(r' - \Gamma(\Gamma^* r' + D_r r)) = r \oplus V_{\mathfrak{R}'} r'. \end{aligned}$$

This finishes the proof of the lemma.

Let now  $T', T, A \in \mathcal{J}(T', T)$  be arbitrary contractions with some fixed minimal isometric dilations  $U, U'$  of  $T, T'$ . For an  $A$ -choice sequence  $\{\Gamma_n\}_1^\infty$  we set

$$(4.3)_n \quad \gamma_n(\Gamma_1, \Gamma_2, \dots, \Gamma_n) \quad (= \gamma_n) = \omega(\Gamma_1) \Gamma_{n+1} \quad (1 \leq n < \infty).$$

Since (see Definition 3.1)

$$\mathfrak{D}_{r_1^*} \supset \mathfrak{D}_{r_2^*} \supset \dots \supset \mathfrak{D}_{r_n^*} \supset \dots,$$

definition (4.3)<sub>n</sub> ( $1 \leq n < \infty$ ) makes sense.

**Lemma 4.3.** *The mapping*

$$(4.4) \quad \{\Gamma_n\}_1^\infty \rightarrow \{\Gamma_1, \{\gamma_n(\Gamma_1, \dots, \Gamma_n)\}_1^\infty\}$$

*establishes a one-to-one correspondence between all  $A$ -choice sequences and all pairs formed by a contraction  $\Gamma: \mathfrak{R}_A \rightarrow \mathfrak{R}'_A$  (considered as belonging to  $\mathcal{J}(O_{\mathfrak{R}'}, O_{\mathfrak{R}})$ ) and a  $\Gamma$ -choice sequence.*

**Proof.** Let  $\{\Gamma_n\}_1^\infty$  be an  $A$ -choice sequence and let  $\{\gamma_n\}_1^\infty$  be the sequence yielded by (4.3)<sub>n</sub> ( $1 \leq n < \infty$ ). It is obvious that, by virtue of Lemma 4.2, we have

$$(4.5)_n \quad D_{\gamma_n} = D_{\Gamma_{n+1}} \quad (1 \leq n < \infty)$$

and, using also (4.1),

$$(4.6)_1 \quad \gamma_1: \mathfrak{R}_{r_1} = \mathfrak{D}_{r_1} \rightarrow \omega(\Gamma_1) \mathfrak{D}_{r_1^*} = \mathfrak{R}'_{r_1},$$

where, as already indicated above,  $\Gamma_1$  is regarded as belonging to  $\mathcal{J}(O_{\mathfrak{R}'}, O_{\mathfrak{R}})$ ; moreover, we also have

$$(4.7)_n \quad D_{\gamma_n}^2 = \omega(\Gamma_1) D_{\Gamma_{n+1}}^2 \omega(\Gamma_1)^*, \quad D_{\gamma_n}^* \omega(\Gamma_1) = \omega(\Gamma_1) D_{\Gamma_{n+1}}^* \quad (1 \leq n < \infty).$$

From (4.5)<sub>n</sub> and (4.7)<sub>n</sub> we infer readily that

$$(4.8)_n \quad \mathfrak{D}_{\gamma_n} = \mathfrak{D}_{\Gamma_{n+1}}, \quad \mathfrak{D}_{\gamma_n}^* = \omega(\Gamma_1) \mathfrak{D}_{\Gamma_{n+1}}^* \quad (1 \leq n < \infty).$$

Consequently,  $\gamma_{n+1}$  is a contraction from  $\mathfrak{D}_{\gamma_n}$  to  $\mathfrak{D}_{\gamma_n}^*$  ( $1 \leq n < \infty$ ). Together with (4.6)<sub>1</sub> this shows that  $\{\gamma_n\}_1^\infty$  is a  $\Gamma_1$ -choice sequence. If we are given now a pair  $\{\Gamma_1, \{\gamma_n\}_1^\infty\}$  formed by a contraction  $\Gamma_1: \mathfrak{R}_A \rightarrow \mathfrak{R}'_A$  (regarded as belonging to  $\mathcal{J}(O_{\mathfrak{R}'}, O_{\mathfrak{R}})$ ) and a  $\Gamma_1$ -choice sequence  $\{\gamma_n\}_1^\infty$ , then there may exist only one  $A$ -choice sequence  $\{\Gamma_n\}_1^\infty$  which is mapped by (4.4) onto our given pair, namely that given by the formula

$$(4.9)_n \quad \Gamma_{n+1} = \omega(\Gamma_1)^* \gamma_n.$$

It is now easy to infer that if  $\Gamma_{n+1}$  ( $1 \leq n < \infty$ ) are actually defined by (4.9)<sub>n</sub>, then (4.5)<sub>n</sub>, (4.7)<sub>n</sub> and consequently (4.8)<sub>n</sub> are also satisfied for  $n \geq 1$ ; it obviously follows that  $\{\Gamma_n\}_1^\infty$  is an  $A$ -choice sequence. This concludes the proof.

We are now in state to formulate the main result of this section. To this aim let  $T, T', A \in \mathcal{J}(T', T)$  (as well as  $U$  and  $U'$ ) be as above. Let  $A_\infty$  be a CID of  $A$  and let

$$\begin{aligned} A_2: A_\infty &\rightarrow \{C_n\}_1^\infty, & A_3: \{C_n\}_1^\infty &\rightarrow \{\Gamma_n\}_1^\infty, & A_4: \{\Gamma_n\}_1^\infty &\rightarrow \{\Gamma_1, \{\gamma_n\}_1^\infty\}, \\ A_5: \{\Gamma_1, \{\gamma_n\}_1^\infty\} &\rightarrow \{\mathfrak{R}_A, \mathfrak{R}'_A, \Gamma(z)\} \end{aligned}$$

respectively be  $(A_2)$  the inverse mapping of that given in Proposition 2.2,  $(A_3)(A_4)$  the mappings given by Proposition 3.1 and Lemma 4.3,  $(A_5)$  the inverse mapping of that given by Lemma 4.1. Then, the bijectivity property of these mappings directly yields the

**Proposition 4.1.** *The mapping  $A_1 = A_5 \circ A_4 \circ A_3 \circ A_2$  establishes a one-to-one correspondence between the CID's of  $A$  and the contractive analytic  $L(\mathfrak{R}_A, \mathfrak{R}'_A)$ -valued functions.*

**Remark 4.2.** The uniqueness theorem for CID's given in [6] is a direct corollary of Proposition 4.1. Indeed, by virtue of this proposition, there exists a unique CID of  $A$ , if and only if there exists only one contractive analytic function  $\{\mathfrak{R}_A, \mathfrak{R}'_A, \Gamma(z)\}$ . Obviously this happens if and only if at least one of the spaces  $\mathfrak{R}_A$  or  $\mathfrak{R}'_A$  reduces to  $\{0\}$ , i.e. (see [16], Ch. VII) if at least one of the factorizations  $A \cdot T$  and  $T' \cdot A$  is regular.

Let us present a particular case which might be instructive. On this purpose, we shall denote by  $i_{\mathfrak{Q}'}$ , the natural isometric identification of  $\mathfrak{Q}'$  with the subspace  $\{0\} \oplus \mathfrak{Q}'$  of  $\mathfrak{D}_A \oplus \mathfrak{Q}'$  and by  $P_{\mathfrak{Q}_*}$  the orthogonal projection of  $\mathfrak{R}$  onto  $\mathfrak{Q}_* = ((I - UT^*)\mathfrak{H})^\perp$ , where the notation is, as usual, that of Sec. 1. Also let us firstly give the following

**Lemma 4.4.** *The operators  $i_{\mathfrak{Q}'}^* | \mathfrak{R}'_A: \mathfrak{R}'_A \rightarrow \mathfrak{Q}'$ ,  $P_{\mathfrak{Q}_*} | \mathfrak{R}_A: \mathfrak{R}_A \rightarrow \mathfrak{Q}_*$  are injective.*

**Proof.** Let  $P_{\mathfrak{Q}_*} r = 0$ ,  $r \in \mathfrak{R}_A$  or equivalently  $r = U h_1$  for some  $h_1 \in \mathfrak{H}$ , and  $(T^* D_A P + U^*(I - P))r = 0$ . Then

$$(4.10) \quad T^* D_A T h_1 + (I - T^* T) h_1 = 0, \quad h_1 = T^*(I - D_A) T h_1.$$

But  $0 \leq I - D_A \leq I$  implies

$$\|T h_1\|^2 \leq \|h_1\|^2 = (T^*(I - D_A) T h_1, h_1) = \|(I - D_A)^{1/2} T h_1\|^2 \leq \|T h_1\|^2,$$

whence

$$(4.11) \quad (I - D_A)^{1/2} T h_1 = T h_1 \quad \text{and} \quad D_A T h_1 = 0.$$

From (4.10) it follows  $\|(U - T) h_1\|^2 = ((1 - T^* T) h_1, h_1) = 0$ , whence

$$(4.12) \quad r = U h_1 = T h_1 \in \mathfrak{H}.$$

Since  $r \in \mathfrak{D}_A$ , from (4.11) and (4.12) we infer that  $r=0$ . This proves the injectivity of  $P_{\mathfrak{L}_*}|\mathfrak{R}_A$ . Concerning the injectivity of  $i_{\mathfrak{L}'}^*|\mathfrak{R}'_A$ , we notice, firstly, that if  $r' \in \mathfrak{R}'_A$ ,  $i_{\mathfrak{L}'}^*r'=0$ , then  $r'=d \oplus 0$  with some  $d \in \mathfrak{D}_A$ , and secondly, that  $(r', D_A h \oplus (U' - T')Ah) = 0$  ( $h \in H$ ) implies  $D_A d = 0$ ,  $d = 0$ , thus  $i_{\mathfrak{L}'}^*|\mathfrak{R}'_A$  is also injective. Thus the lemma is proved.

By virtue of the preceding lemma and of [16], Ch. II, Sec. I we have

$$\dim \mathfrak{R}_A \leq \delta_{T^*} \stackrel{\text{def}}{=} \text{rank } D_{T^*} = \dim \mathfrak{L}_*, \quad \dim \mathfrak{R}'_A \leq \delta_{T'} \stackrel{\text{def}}{=} \text{rank } D_{T'} = \dim \mathfrak{L}'.$$

Therefore, from Lemma 4.4 and Proposition 4.1 we can now readily obtain the following

**Corollary 4.1.** *Assume that, within the frame of Proposition 4.1, we have  $\delta_{T^*} = \delta_{T'} = 1$ . Then either the set of all CID's of  $A$  is a singleton or it is in a one-to-one correspondence (explicitly given by  $\Lambda_1$ ) with the unit ball of  $H^\infty$  (i.e. the set of all complex-valued analytic functions  $u(z)$  on the unit disk  $D = \{z: |z| < 1\}$  such that  $|u(z)| \leq 1$  for all  $z \in D$ ).*

It is plain that in this corollary the first case occurs if  $\min(\dim \mathfrak{R}_A, \dim \mathfrak{R}'_A) = 0$  (see Remark 4.2), while the second one if  $\dim \mathfrak{R}_A = \dim \mathfrak{R}'_A = 1$ .

5. We shall now apply Proposition 4.1 to the labelling of all classes of isomorphic Ando dilations. To be more precise, for a pair  $\{T_1, T_2\}$  of some fixed commuting contractions on some Hilbert space  $\mathfrak{H}$ , there always exists (as shown in a celebrated short note by ANDO [5]) a pair  $\{U_1, U_2\}$  of commuting isometric operators on some Hilbert space  $\mathfrak{K}$  containing  $\mathfrak{H}$  as a (closed linear) subspace and such that

$$(5.1) \quad PU_1^{n_1}U_2^{n_2}|\mathfrak{H} = T_1^{n_1}T_2^{n_2} \quad (n_1, n_2 = 0, 1, 2, \dots),$$

where  $P$  denotes the orthogonal projection of  $\mathfrak{K}$  onto  $\mathfrak{H}$ . Obviously we can and shall also suppose that

$$(5.2) \quad \mathfrak{K} = \bigvee_{n_1, n_2 \geq 0} U_1^{n_1}U_2^{n_2}\mathfrak{H}.$$

Any such pair  $\{U_1, U_2\}$  will be called an *Ando dilation* of  $\{T_1, T_2\}$ . Two Ando dilations  $\{U_1, U_2\}$ ,  $\{U'_1, U'_2\}$  are called isomorphic if there exists a unitary operator  $W$  from the space  $\mathfrak{K}$ , on which operate  $U_1$  and  $U_2$ , to the space  $\mathfrak{K}'$  on which operate  $U'_1$  and  $U'_2$ , such that

$$(5.3) \quad WU_j = U'_jW \quad (j = 1, 2), \quad W|\mathfrak{H} = I_{\mathfrak{H}}.$$

Let now  $U$  on  $\mathfrak{K}$  be a fixed minimal isometric dilation of  $T = T_1$ . Obviously any Ando dilation  $\{U'_1, U'_2\}$  is isomorphic with some Ando dilation  $\{U_1, U_2\}$  operating on a space  $\mathfrak{K}$  containing  $\mathfrak{K}$  (as closed linear subspace), and such that

$$(5.4) \quad U_1|_{\mathfrak{K}} = U.$$

Let  $\{U'_1, U'_2\}$  be another such Ando dilation, isomorphic "by  $W$ " to  $\{U_1, U_2\}$ . Then by virtue of (5.3) we have  $WU^n h = WU_1^n h = U_1'^n W h = U_1'^n h = U^n h$  for all  $h \in \mathfrak{H}$ ,  $n=0, 1, 2, \dots$ ; therefore

$$(5.5) \quad W|K = I_K.$$

By virtue of this discussion, we can and shall consider from now on Ando dilations satisfying (5.4). With this convened, we state the following

**Lemma 5.1.** *For  $T=T_1$  and  $A=T_2$ , the formula*

$$(5.6) \quad \hat{A} = P_K U_2|K$$

(where  $P_K$  denotes the orthogonal projection of  $\mathfrak{R}$  onto  $K$ ) establishes a one-to-one correspondence between all classes of isomorphic Ando dilations of  $\{T_1, T_2\}$  and all CID's of  $A$ .

**Proof.** First we remark that

$$(5.7) \quad U_1^* K \subset K,$$

i.e. that  $K$  is reducing  $U_1$ . This was, for instance, proven in [13]. For the sake of completeness let us sketch the proof. On this purpose we infer easily from (5.1) and (5.2) that

$$(5.8) \quad P U_1 = T_1 P = T P, \quad P U_2 = T_2 P = A P;$$

from the first relation (5.8) it follows that

$$(5.9) \quad U_1^* \mathfrak{H} = T_1^* = T^*,$$

whence, for  $h \in \mathfrak{H}$ ,

$$U_1^* U^n h = T^* h \quad \text{if } n=0, \quad \text{and} \quad U_1^* U^n h = U^{n-1} h \quad \text{if } n=1, 2, \dots,$$

so that, since these  $U^n h$ 's span  $K$ , (5.7) is true. We conclude thus that

$$(5.10) \quad P_K U_1 = U_1 P_K.$$

Now the fact that for a given  $\{U_1, U_2\}$  formula (5.6) defines a CID  $\hat{A}$  of  $A$  can be easily obtained from (5.10) and the second relation (5.8). Moreover if  $\{U'_1, U'_2\}$  is another Ando dilation of  $\{T_1, T_2\}$ , isomorphic (by  $W$ ) to  $\{U_1, U_2\}$ , then  $W P_K = P_K = P_K W$ , so that

$$P_K U'_2|K = P_K W U_2|K = P_K U_2|K = \hat{A}.$$

Thus we can conclude that (5.6) defines a mapping from the classes of isomorphic Ando dilations of  $\{T_1, T_2\}$  to the set of the CID's of  $A$ . Let now  $\hat{A}$  be a CID of  $A$ . Let  $U_2$  on  $\mathfrak{K}$  be a minimal isometric dilation of  $\hat{A}$ . Since  $U$  is an isometric operator commuting with  $A$  it has a unique CID (as operator in  $\mathfrak{K}$  commuting with  $U_2$ ; namely consider in Remark 4.2 the case when  $A$  is isometric and observe that in this particular case we have  $\mathfrak{R}_A = \{0\}$ ), which we shall denote by  $U_1$ . The pair  $\{U_1, U_2\}$  is an Ando dilation of  $\{T_1, T_2\}$  satisfying the property (5.6). Indeed, (5.6) is satisfied by the very definition of  $U_2$ , while

$$\begin{aligned} PU_1^{n_1} U_2^{n_2} &= PP_K U_1^{n_1} U_2^{n_2} = PU_1^{n_1} P_K U_2^{n_2} = \\ &= PU_1^{n_1} \hat{A}^{n_2} P_K = T_1^{n_1} P \hat{A}^{n_2} P_K = T_1^{n_1} A^{n_2} PP_K = T_1^{n_1} A^{n_2} P = T_1^{n_1} T_2^{n_2} P \end{aligned}$$

for  $n_1, n_2 = 0, 1, 2, \dots$ . Moreover, since  $U$  is isometric, it follows directly that  $U_1|_{\mathfrak{K}} = U$ , whence

$$\begin{aligned} \left\| U_1 \sum_{n=0}^N U_2^n k_n \right\|^2 &= \left\| \sum_{n=0}^N U_2^n U k_n \right\|^2 = \\ &= \sum_{N \geq n \geq m \geq 0} (U_2^{n-m} U k_n, U k_m) + \sum_{0 \leq n < m \leq N} (U k_n, U_2^{m-n} U k_m) = \\ &= \sum_{N \geq n \geq m \geq 0} (\hat{A}^{n-m} U k_n, U k_m) + \sum_{0 \leq n < m \leq N} (U k_n, \hat{A}^{m-n} U k_m) = \\ &= \sum_{N \geq n \geq m \geq 0} (U \hat{A}^{n-m} k_n, U k_m) + \sum_{0 \leq n < m \leq N} (U k_n, U \hat{A}^{m-n} k_m) = \\ &= \sum_{N \geq n \geq m \geq 0} (\hat{A}^{n-m} k_n, k_m) + \sum_{0 \leq n < m \leq N} (k_n, \hat{A} k_m) = \left\| \sum_{n=0}^N U_2^n k_n \right\|^2 \end{aligned}$$

for all  $k_1, k_2, \dots, k_N \in \mathfrak{K}$ ,  $N = 0, 1, \dots$ . Therefore  $U_1$  is indeed isometric.

Finally, the fact that relation (5.2) is also satisfied, follows from

$$\bigvee_{n_1, n_2 \geq 0} U_1^{n_1} U_2^{n_2} \mathfrak{H} = \bigvee_{n_2 \geq 0} U_2^{n_2} \bigvee_{n_1 \geq 0} U_1^{n_1} \mathfrak{H} = \bigvee_{n_2 \geq 0} U_2^{n_2} \bigvee_{n_1 \geq 0} U^{n_1} \mathfrak{H} = \bigvee_{n_2 \geq 0} U_2^{n_2} \mathfrak{K} = \mathfrak{K}$$

because  $U$  and  $U_2$  are minimal isometric dilations of  $T (= T_1)$  and  $\hat{A}$ , respectively.

It remains to prove that the mapping yielded by (5.6), is one-to-one. But this follows at once from the preceding construction, since if  $P_K U'_2|_{\mathfrak{K}} = P_K U_2|_{\mathfrak{K}} (= \hat{A})$  for two Ando dilations  $\{U_1, U_2\}$ ,  $\{U'_1, U'_2\}$ , the isometries  $U_2$  and  $U'_2$  are actually minimal isometric dilations of  $\hat{A}$ , thus isomorphic, say by the unitary operator  $W$ . But then

$$W U_1 U_2^{n_2} k = W U_2^{n_2} U k = U_2'^{n_2} W U k = U_2'^{n_2} U k = U'_1 U_2'^{n_2} k = U'_1 W U_2^{n_2} k$$

for all the elements  $U_2^{n_2} k$  ( $k \in \mathfrak{K}$ ,  $n_2 = 0, 1, 2, \dots$ ). Since these elements span the space on which operate  $U_1$  and  $U_2$ , we infer that (5.3) is valid (of course in the special case satisfying (5.4)), thus  $\{U_1, U_2\}$  and  $\{U'_1, U'_2\}$  are isomorphic.

**Proposition 5.1.** *Let  $\{T_1, T_2\}$  be a pair of commuting contractions on  $\mathfrak{H}$  and let, for  $i=1, j=2$  or  $i=2, j=1$ ,*

$$(5.10) \quad \mathfrak{R}_{ij} = (\mathfrak{D}_{T_i} \oplus \mathfrak{D}_{T_j}) \ominus \{D_{T_i} T_j h \oplus D_{T_j} h : h \in \mathfrak{H}\}.$$

*There exists a one-to-one (explicit) correspondence between all classes of isomorphic Ando dilations of  $\{T_1, T_2\}$  and all the contractive analytic  $L(\mathfrak{R}_{21}, \mathfrak{R}_{12})$ -valued functions.*

**Proof.** We set  $T=T_1$  and  $A=T_2$ . By virtue of Lemma 5.1 and Proposition 4.1 we have an explicit one-to-one correspondence from the classes of isomorphic Ando dilations of  $\{T_1, T_2\}$  and all contractive analytic functions  $\{\mathfrak{R}_A, \mathfrak{R}'_A, B(z)\}$ . Or by virtue of [16], Ch. II, Sec. 1, there exists a unitary (canonical) identification  $\varphi: D_T h \rightarrow (U-T)h$  of  $\mathfrak{D}_T = \mathfrak{D}_{T_1}$  with  $\mathfrak{L}$ . Thus  $\varphi_1 = [I_{D_{T_2}}, \varphi]$  identifies  $\mathfrak{D}_{T_2} \oplus \mathfrak{D}_{T_1}$  to  $\mathfrak{D}_{T_2} \oplus \mathfrak{L}$  and takes  $\mathfrak{R}_{21}$  onto  $\mathfrak{R}_A$ , while  $\varphi'_1 = I_{\mathfrak{D}_{T_2}} \oplus \varphi$  identifies  $\mathfrak{D}_{T_2} \oplus \mathfrak{D}_{T_1}$  to  $\mathfrak{D}_{T_2} \oplus \mathfrak{L}$  and takes  $\mathfrak{R}'_{21} = (\mathfrak{D}_{T_2} \oplus \mathfrak{D}_{T_1}) \ominus \{D_{T_2} h \oplus D_{T_1} T_2 h : h \in \mathfrak{H}\}$  onto  $\mathfrak{R}'_A$ .

Denoting  $\varphi'$  the unitary operator from  $\mathfrak{D}_{T_2} \oplus \mathfrak{D}_{T_1}$  to  $\mathfrak{D}_{T_1} \oplus \mathfrak{D}_{T_2}$  which intertwines the coordinates, we obtain by  $A(z) = \psi \varphi'^* B(z) \varphi_1 |R_{21}|$  ( $|z| < 1$ ) the mapping yielding the one-to-one correspondence between the set of all contractive analytic function  $\{\mathfrak{R}_A, \mathfrak{R}'_A, B(z)\}$  and that of those of the form  $\{\mathfrak{R}_{21}, \mathfrak{R}_{12}, A(z)\}$ . This plainly concludes the proof.

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